Hybrid curves and their moduli spaces

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Topology and arithmetic on the moduli space of curves

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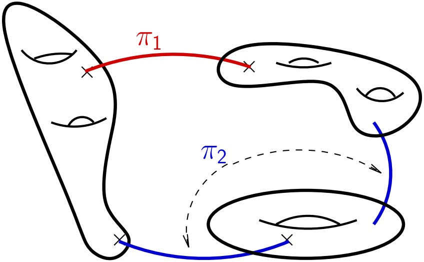
Hybrid curves



A **hybrid curve** $\mathcal{C}$ consists of

- a finite **graph** $G = (V, E)$
- for each vertex v: a compact **Riemann surface** C_v
- on each C_v, **attachment points** p_v^e, $e \sim v$, for interval edges
- an **ordered partition** $\pi = (\pi_1, \ldots, \pi_r)$ of the edge set E
- an **edge length function** $\ell \colon E \to (0, \infty)$ (with $\sum_{e \in \pi_j} \ell(e) = 1 \, \forall j$)

Hybrid curve = algebraic curve + tropical curve

Motivation

$\bar{\mathscr{M}}_g$... Deligne–Mumford compactification of $\mathscr{M}_g$

Let X_S be an **interesting object** associated to Riemann surfaces $S \in \mathscr{M}_g$ (e.g., invariant, metric on S, solution to differential equation on S, ...)

Question

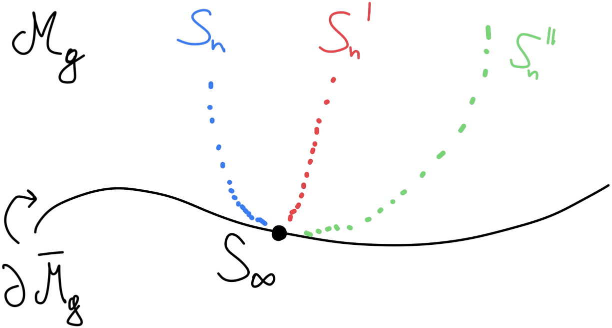
Suppose $S \in \mathscr{M}_g$ converges to a stable Riemann surface $S_\infty \in \bar{\mathscr{M}}_g \setminus \mathscr{M}_g$.

What happens to X_S?

Ideally, one can extend X_S continuously to $\bar{\mathscr{M}}_g$, i.e.

(*i*) define an analogous object X_{S_∞} on stable Riemann surfaces S_∞
(*ii*) prove that $X_S \to X_{S_\infty}$ as $S \to S_\infty$

Sequences approaching a boundary point in $\overline{\mathscr{M}}_g$



Certain interesting objects X_S **can't be extended continuously** to $\overline{\mathscr{M}}_g$, since their limit depends on the "way of approaching S_∞"!

Canonical (a.k.a. Arakelov–Bergman) measure

Ω ... space of **holomorphic 1-forms** ω on S, equipped with **inner product**

$$\langle \omega, \eta \rangle_S = \frac{i}{2} \int_S \omega \wedge \overline{\eta}, \qquad \omega, \eta \in \Omega.$$

Definition

Fix an ON-basis $(\omega_k)_k$ of Ω. The **canonical measure** μ_{can} on S is

$$\mu_S := \frac{i}{2} \sum_{k=1}^{g} \omega_k \wedge \overline{\omega_k}.$$

Relationship to the polarized Jacobian

μ_S is the Riemannian measure of the **canonical metric** φ_S on S:

- $\langle \cdot, \cdot \rangle_S$ induces a metric φ on Jacobian $\mathrm{Jac}(S) = \Omega^* / H_1(S, \mathbb{Z})$
- φ_S is the pull-back of φ via Abel–Jacobi map

$$AJ \colon S \hookrightarrow \mathrm{Jac}(S).$$

Applications of the canonical measure

The canonical measure μ_S gives rise to other interesting objects:

- E.g., the **Arakelov Green function** $g_S \colon S \times S \to \mathbb{R}$ is the solution to the distributional Poisson equations

$$\frac{1}{\pi i}\partial_z \partial_{\bar{z}} g_S(x,\cdot) = \mu_S - g\cdot\delta_x, \qquad \int_S g_S(x,y)\, d\mu_S(y) = 0, \qquad x \in S.$$

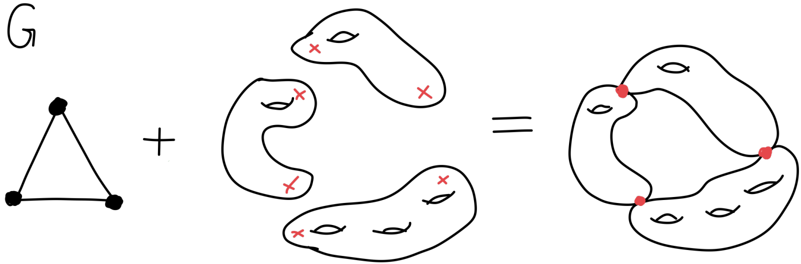
- E.g., interesting **invariants of a Riemann surface**

  δ-invariant: Faltings, *Calculus on arithmetic surfaces*, Ann. Math. (1984)

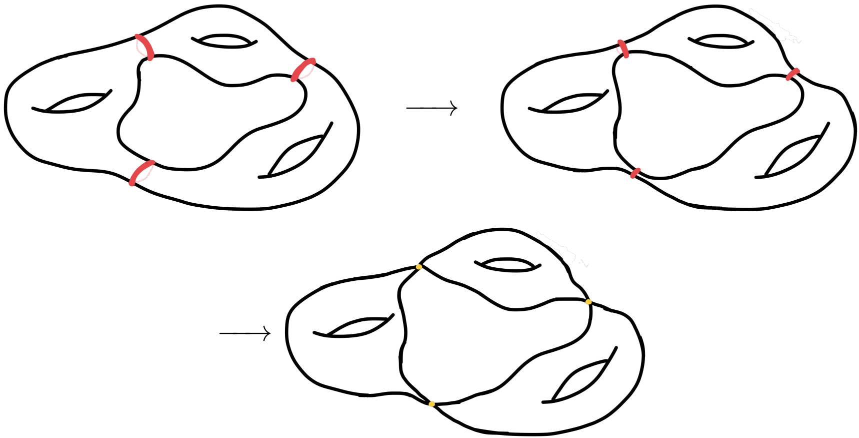
Many authors studied μ_S and related objects on degenerating RS's!

However, they are examples of objects X_S, which cannot be extended continuously to $\overline{\mathscr{M}}_g$!

Let S_∞ be a **stable Riemann surface** with graph $G = (V, E)$.



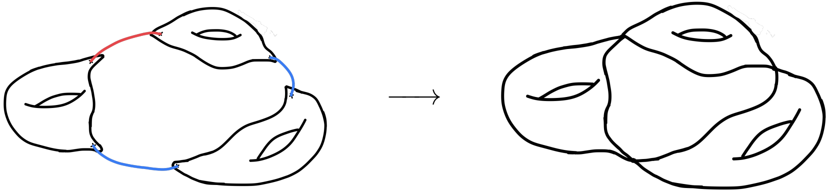
Degenerating Riemann surfaces



Constructing a “larger” compactifiation

Plan: find other compactification of $\mathscr{M}_g \Rightarrow$ moduli space of hybrid curves

To each hybrid curve $\mathcal{C}$, one assigns a stable Riemann surface $S_\infty(\mathcal{C})$:



There are infinitely many hybrid curves for each stable RS $S_\infty \in \bar{\mathscr{M}}_g \setminus \mathscr{M}_g$!

Idea of construction:

“Replace each stable Riemann surface $S_\infty \in \bar{\mathscr{M}}_g \setminus \mathscr{M}_g$
by its infinitely many hybrid curves”

Convergence of Riemann surfaces to hybrid curves

Definition

Let $\mathcal{C}$ be a hybrid curve with: underlying stable RS $S_\infty(\mathcal{C})$;
ℓ edge length function; $\pi = (\pi_1, \dots, \pi_r)$ ordered partition;

A sequence $(S_n)_n \subset \mathscr{M}_g$ **converges** to $\mathcal{C}$ if, as $n \to \infty$,

(i) $(S_n)_n$ converges to $S_\infty(\mathcal{C})$ in $\overline{\mathscr{M}}_g$

(ii) cycles for edges in π_j shrink "infinitely faster" than cycles of π_{j+1},

(iii) the "relative shrinking speed" of cycles for edges in the same set π_j is captured by the edge lengths.

(ii) More precisely: for all sets π_j in $\pi = (\pi_1, \dots, \pi_r)$:

$$\frac{|\log|z_e(S_n)||}{|\log|z_{e'}(S_n)||} \to \infty \qquad e \in \pi_j, e' \in \pi_{j+1}.$$

(iii) More precisely: for all sets π_j in $\pi = (\pi_1, \dots, \pi_r)$:

$$\frac{|\log|z_e(S_n)||}{|\log|z_{e'}(S_n)||} \to \frac{\ell(e)}{\ell(e')}, \qquad \text{for all } e, e' \in \pi_j.$$

The moduli space of hybrid curves

Theorem (Amini–N; 2021)

The moduli space $\mathscr{M}_g^{hyb}$ of hybrid curves of genus g **compactifies** $\mathscr{M}_g$.

$\mathscr{M}_g^{hyb}$ **refines the DM–compactification** $\bar{\mathscr{M}}_g$ in the sense that

$$\mathscr{M}_g^{hyb} \to \bar{\mathscr{M}}_g, \qquad \mathcal{C} \mapsto S_\infty(\mathcal{C}),$$

is a continuous map.

Applications

Q: Understand objects X_S on degenerating Riemann surfaces S?

Strategy:

(*i*) define a suitable object $X_{\mathcal{C}}$ on **hybrid curves** $\mathcal{C}$

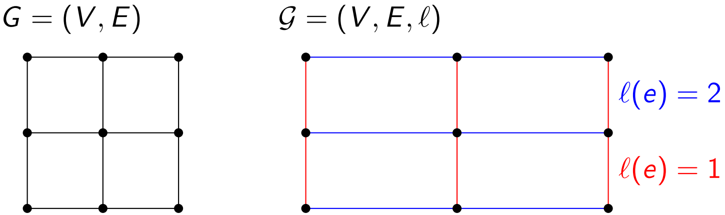
(*ii*) prove that, as $S \to \mathcal{C}$ in $\mathscr{M}_g^{hyb}$, one can **describe** X_S **in terms of** $X_{\mathcal{C}}$

We did this for a couple of interesting objects X_S.

Metric graphs

Definition

A **metric graph** $\mathcal{G} = (V, E, \ell)$ is a (finite) graph $G = (V, E)$ together with an **edge length function** $\ell \colon E \to (0, +\infty)$.



Canonical measure on metric graphs (Zhang; Invent. Math.,'92)

Let $\mathcal{G} = (V, E, \ell)$ be a metric graph. Fix an orientation on the edge set E.

- A **harmonic one-form** on $\mathcal{G}$ is a map $\omega\colon E \to \mathbb{R}$ satisfying

$$\sum_{\text{incoming edges at } v} \omega(e) = \sum_{\text{outgoing edges at } v} \omega(e), \qquad v \in V.$$

- The **space of harmonic one-forms** $H^1(\mathcal{G})$ has the inner product

$$\langle \omega, \eta \rangle_{\mathcal{G}} = \sum_{e \in E} \ell(e)\, \omega(e) \eta(e), \qquad \omega, \eta \in H^1(\mathcal{G}).$$

The **canonical measure** on $\mathcal{G}$ is the edgewise weighted Lebesgue measure

$$\mu_{\mathcal{G}} := \sum_{e \in E} \mu_e \cdot \lambda_e$$

where λ_e is the Lebesgue measure on the interval edge $e = [0, \ell(e)]$ and

$$\mu_e = \frac{1}{\ell(e)} \sum_k \omega_k(e)^2, \quad \text{with } (\omega_k)_k \text{ an ON-basis of } H^1(\mathcal{G}).$$

Graded minors of a hybrid curve

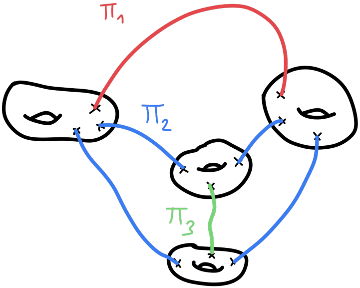
Definition

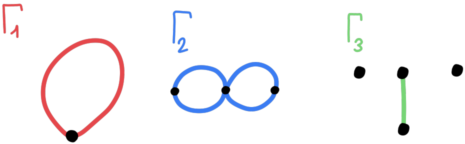
For $j = 1, \ldots, r$, the j-th **graded minor** of $\mathcal{C}$ is the metric graph Γ_j with edge set π_j and length function $\ell|_{\pi_j}$ obtained by:

(*) contracting all Riemann surface components C_v to points

(*) removing all intervals for edges in $\pi_1 \cup \pi_2 \cup \cdots \cup \pi_{j-1}$

(*) contracting all intervals for edges in $\pi_{j+1} \cup \pi_{j+2} \cup \cdots \cup \pi_r$





Hybrid canonical measure and convergence result

Definition

Let $\mathcal{C}$ be a hybrid curve. The **canonical measure** on $\mathcal{C}$ is

$$\mu_{\mathcal{C}} := \sum_{v \in V} \mu_{C_v} + \sum_{j=1}^{r} \mu_{\Gamma^j}.$$

(C_v's... Riemann surface components of $\mathcal{C}$; $\Gamma_1, \ldots \Gamma_r$... graded minors)

Theorem (Amini–N., 2021)

Let $(S_n)_n \subset \mathscr{M}_g$ be a sequence of Riemann surfaces and $\mathcal{C}$ a hybrid curve.

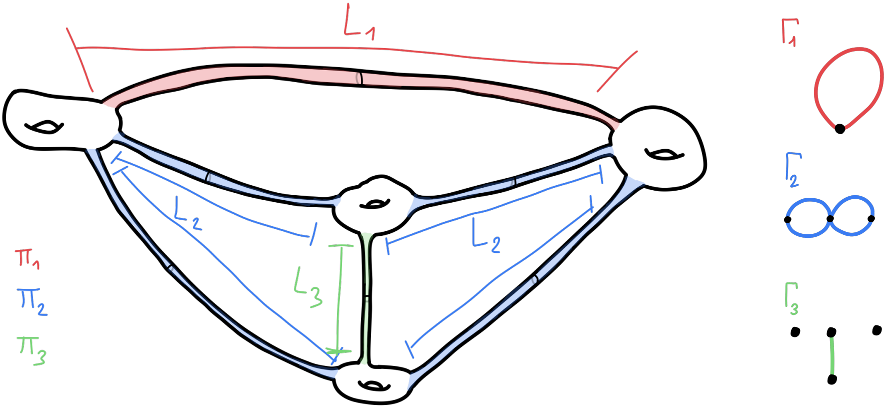
$$\text{If } S_n \to \mathcal{C}, \text{ then } \mu_{S_n} \to \mu_{\mathcal{C}} \qquad \text{(in a weak sense).}$$

Altogether, "**canonical measures vary continuously**" over $\mathscr{M}_g^{hyb}$.

Canonical metric on degenerating Riemann surfaces

μ_S is the measure associated to the **canonical metric** φ_S on S

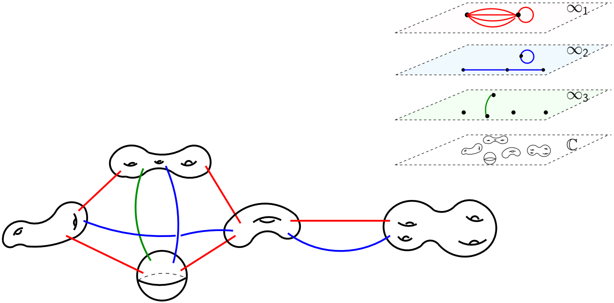
If S converges to a hybrid curve, φ_S exhibits a **multi-scale degeneration**:



Here $L_j = \sum_{e \in \pi_j} |\log |t_e||$ and, in particular, $L_1 \gg L_2 \gg ... \gg Lr$

Note: this allows to "see the graded minors"!

Another perspective on hybrid curves



Actually, one should imagine a hybrid curve as a "multi-scale object", with pieces $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ and $\bigsqcup_v C_v$:

- first graded minor $\Gamma_1 \to$ "dominant scale"
- second graded minor $\Gamma_2 \to$ "2nd dominant scale"
- ...
- Riemann surfaces $\bigsqcup_v C_v \to$ "constant order scale"

Poisson equation on degenerating Riemann surfaces

Let $S \in \mathscr{M}_g$ and ν_S a measure ν_S on S. Consider the **Poisson equation**

$$\Delta_S f = \nu_S, \qquad \Delta_S := \frac{1}{\pi i}\partial_z \partial_{\bar{z}}.$$

Q: What happens to the **solution** f_S, when S degenerates?

Hybrid function and hybrid Laplacian

Definition

$\mathcal{C}$ hybrid curve with graded minors $\Gamma_1, \ldots, \Gamma_r$ and RSs C_v, $v \in V$:
A **hybrid function** on $\mathcal{C}$ is a **tuple** $\mathbf{f} = (f_1, \ldots, f_r, f_{\mathbb{C}})$, where

- for each $j = 1, \ldots, r$: $f_j \colon \Gamma^j \to \mathbb{C}$ is a function on Γ^j
- $f_{\mathbb{C}} \colon \bigsqcup_v C_v \to \mathbb{C}$ is a function on $\bigsqcup_v C_v$

We "mix Laplacians on Riemann surfaces / graphs" to define a **Laplacian**

$$\Delta_{\mathcal{C}}(\mathbf{f}) \in \{\nu | \, \nu \text{ Borel measure on } \mathcal{C}\}.$$

E.g., for $r = 1$ (the trivial ordered partition $\pi = (E)$):

$$\Delta_{\mathcal{C}}(f_1, f_{\mathbb{C}}) := \sum_{e \in E} -(f_1|_e)'' \lambda_e + \sum_{v \in C_v} \Delta_{C_v}(f_{\mathbb{C}})|_{C_v} + \sum_{e \sim v} \partial_e f_1(v) \delta_{p_v^e}.$$

(Here: p_v^e, $e \sim v$ are the "attachment points" of intervals on C_v)

Asymptotics of solutions to Poisson equation

Theorem (Amini–N., 2022)

Let $S \in \mathscr{M}_g$ and $\Delta_S(f_S) = \nu_S$ a Poisson equation with solution f_S.
Let $\mathcal{C}$ be a hybrid curve and $\Delta_{\mathcal{C}}(\mathbf{f}) = \nu_{\mathcal{C}}$ a Poisson equation on $\mathcal{C}$ with solution $\mathbf{f} = (f_1, \ldots, f_r, f_{\mathbb{C}})$.

If $S \to \mathcal{C}$ and $\nu_S \to \mu_{\mathcal{C}}$ (plus technical assumptions), then

$$f_S \approx L_1(S) f_1 + L_2(S) f_2 + \ldots L_r(S) f_r + f_{\mathbb{C}} + o(1),$$

where $L_j = \sum_{e \in \pi_j} |\log|t_e||$. In particular $L_1(S) \gg \cdots \gg L_r(S) \to \infty$.

In particular, we obtain **asymptotics for the Arakelov–Green function** g_S on degenerating Riemann surfaces!

prior results: Wentworth (1991), de Jong (2019) and Faltings (2021)

Divisors and meromorphic functions on hybrid curves

Let $\mathcal{C}$ be a hybrid curve.

A **divisor** D on $\mathcal{C}$ is a finite integer combination of points on $\mathcal{C}$.

A **meromorphic function** on $\mathcal{C}$ is a hybrid function $\mathbf{f} = (f_1, \dots, f_r, f_\mathbb{C})$ s.t.

- $f_\mathbb{C} \colon \bigsqcup_{v \in V} C_v \to \mathbb{C}$ is meromorphic on each Riemann surface C_v
- for all $j = 1, \dots, r$, the function $f_j \colon \Gamma^j \to \mathbb{C}$ is meromorphic on Γ^j, i.e. continuous, piecewise linear with integer slopes

Poincaré–Lelong formula on Riemann surfaces S:

$$\mathrm{div}(f) = \Delta_S(-\log|f|), \qquad f \text{ meromorphic on } S.$$

The **principal divisor** of a meromorphic function $\mathbf{f}$ on $\mathcal{C}$ is

$$\mathrm{div}(\mathbf{f}) = \Delta_\mathcal{C}(f_1, \dots, f_r, -\log|f_\mathbb{C}|) = \mathrm{div}_{\Gamma^1}(f_1) + \dots \mathrm{div}_{\Gamma^r}(f_r) + \mathrm{div}_{\bigsqcup_v C_v}(f_\mathbb{C}).$$

Hybrid meromorphic functions & principal divisors as limits

Theorem (Amini–N, 2024+)

"Limits of principal divisors are hybrid principal divisors."

I.e., let $S \in \mathscr{M}_g$ equipped with a principal divisor $D_S \in \mathrm{Prin}(S)$. Suppose S converges to a hybrid curve $\mathcal{C}$ and D_S converges to a divisor $D_{\mathcal{C}}$ on $\mathcal{C}$.

Then, the limiting divisor $D_{\mathcal{C}}$ is **principal**.

For the corresponding meromorphic functions f_S on S and $\mathbf{f} = (f_j)_j$ on $\mathcal{C}$:

$$|f_S| = \lambda_1^{f_1 + o(1)} \cdot \lambda_2^{f_2 + o(1)} \cdot \lambda_r^{f_r + o(1)} \cdot |f_{\mathbb{C}}|(1 + o(1)),$$

where $\lambda_j = \prod_{e \in \pi_j} |t_e|$ for $j = 1, \ldots, r$.

(Here, f_S and $\mathbf{f}$ are chosen suitably normalized!!!):

Picard group and Jacobian of a hybrid curve

Definition

Let $\mathcal{C}$ be a hybrid curve. Define

$$\mathrm{Pic}(\mathcal{C}) := \mathrm{Div}^0(\mathcal{C}) / \mathrm{Prin}(\mathcal{C})\ldots$$ **Picard group**

(degree zero divisors modulo principal divisors)

$$\mathrm{Jac}(\mathcal{C}) := H_1(\mathcal{C}, \mathbb{R}) / H_1(\mathcal{C}, \mathbb{Z})\ldots$$ **Jacobian** of $\mathcal{C}$

- $\mathrm{Pic}(\mathcal{C})$ and $\mathrm{Jac}(\mathcal{C})$ are **abelian groups** endowed with **filtrations**:

$$\mathrm{Pic}(\mathcal{C}) = P^1 \supset P^2 \supset \ldots P^r \supset P^{\mathbb{C}}$$
$$\mathrm{Jac}(\mathcal{C}) = J^1 \supset J^2 \supset \ldots J^r \supset J_{\mathbb{C}}$$

P^j, J^j... "allow only points and cycles not touching edges in $\pi_1 \cup \ldots \pi_{j-1}$"

- Quotients are the **Jacobians and Picard groups of graded minors**:

$$P^j / P^{j+1} \cong \mathrm{Pic}(\Gamma^j), \qquad J^j / J^{j+1} \cong \mathrm{Jac}(\Gamma^j).$$

Hybrid Abel–Jacobi map

Theorem (Amini–N., 2024+)

Let $\mathcal{C}$ be a hybrid curve.

- We construct a natural **Abel–Jacobi map**

$$\mathrm{AJ}_{\mathcal{C}}\colon \quad \mathrm{Pic}(\mathcal{C}) \quad \to \quad \mathsf{Jac}(\mathcal{C}),$$

  which is an isomorphism of abelian groups respecting the filtrations.
- The induced maps on quotients are the **Abel–Jacobi maps of graded minors and Riemann surfaces** of $\mathcal{C}$:

$$\mathrm{AJ}_{\mathcal{C}}\colon P^j/P^{j+1} \to J^j/J^{j+1} \quad \cong \quad \mathrm{AJ}_{\Gamma^j}\colon \mathsf{Prin}(\Gamma^j) \to \mathsf{Jac}(\Gamma^j)$$

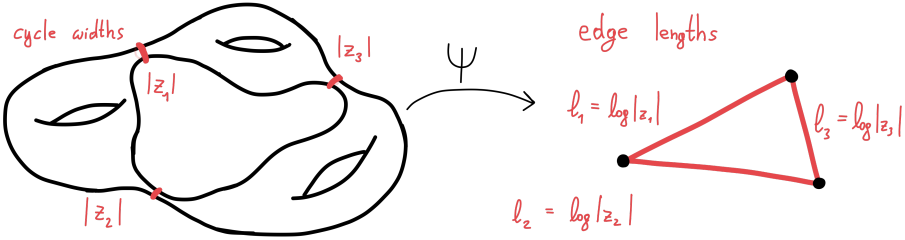
$$\mathrm{AJ}_{\mathcal{C}}\colon P^{\mathbb{C}} \to J^{\mathbb{C}} \quad \cong \quad \bigoplus_v \mathrm{AJ}_{C_v}\colon \bigoplus_v \mathsf{Prin}(C_v) \to \bigoplus_v \mathsf{Jac}(C_v)$$

- If $S \in \mathscr{M}_g$ converges to $\mathcal{C}$ in $\mathscr{M}_g^{hyb}$, then the Abel–Jacobi map AJ_S on S **converges** to $\mathrm{AJ}_{\mathcal{C}}$.

Relationship to compactifications of tropical moduli spaces

$\mathscr{M}_g^{trop}$ … moduli space of augmented genus g metric graph

Close to a stable RS S_∞ of graph G, there is a map $\psi\colon \mathscr{M}_g \to \mathscr{M}_g^{trop}$:



If $S \to S_\infty$ in $\overline{\mathscr{M}}_g$, then $\ell(e) \to \infty$ for all $e \in E$ ("degenerating graphs")

$\mathscr{M}_g^{hyb}$ has an **analog compactification** of $\mathscr{M}_g^{trop}$!
(using metric graphs with "ordered partition of edge set")

Further results: geometry of hybrid curves

Theorem (Amini–N.; 24+)

Suppose that a Riemann surface S degenerates to a hybrid curve $\mathcal{C}$. Let $\mathrm{Jac}(S)$ be the polarized Jacobian of S, viewed as a Riemannian manifold.

Then, in the Gromov–Hausdorff sense and up to proper normalization,

- $\mathrm{Jac}(S)$ **converges to the polarized graph Jacobian** $\mathrm{Jac}(\Gamma_1)$
- The polarized graph Jacobians $\mathrm{Jac}(\Gamma_2), \ldots, \mathrm{Jac}(\Gamma_r)$ and Riemann surface Jacobians $\mathrm{Jac}(C_v)$, $v \in V$, appear as limits of explicit subtori.