TOPOLOGY AND ARITHMETIC OF MODULI SPACES OF CURVES

1. MINICOURSES

Francis Brown – Cohomology of moduli spaces of tropical curves, abelian varieties and the general linear group of integers.

My first lecture aims to give a bird's eye view of the mathematical landscape. I will explain how the areas in the title (and others) are interrelated, survey the main conjectures and theorems in these fields, and explain the role played by graph complexes and the Grothendieck–Teichmüller group.

In particular I would like to mention a string of very recent results, by several different groups of authors, which have completely transformed what we now know about the unstable cohomology of all these spaces.

The remaining lectures will be introductory and focus on one or two key ideas. I will discuss the geometry of the moduli space of tropical abelian varieties, and the important role played by invariant differential forms.

Richard Hain – Topology, Hodge Theory and Motives.

This mini course comprises three relatively independent lectures. The main theme will be how the topology of a variety affects and is affected by motives associated with it.

Lecture I. The Goldman–Turaev Lie bialgebra; is it motivic?

The Goldman–Turaev Lie bialgebra of an oriented 2-manifold X is a Lie bialgebra structure on the free abelian group spanned by the conjugacy classes of its fundamental group. Its structure encodes how isotopy classes of immersed loops on X intersect each other and themselves. When X is a smooth complex curve, a suitable completion of the GT-Lie bialgebra carries a natural mixed Hodge structure. The bracket and cobracket are both morphisms (after a suitable twist). This raises the question of whether (when X is defined over a number field) of whether the bracket and cobracket are motivic and, if so, how they are related to algebraic cycles.

I will define the Goldman bracket and Turaev cobracket, as well as their extensions by Kawazumi and Kuno. This is elementary and beautiful surface topology. I'll survey what is known about the MHS and Galois actions on the completed GT-Lie bialgebra and its connections to the Johnson homomorphism.

Lecture II. The Ceresa cycle, its normal function and its rank

Normal functions are certain holomorphic sections of families of "intermediate jacobians". They were first defined by Griffiths over 40 years ago to study homologically trivial algebraic cycles and were motivated by work of Poincaré. I will define normal functions and explain how they arise from families of homologically trivial cycles.

The genus $g > 2$ Ceresa cycle is a family of homologically trivial cycles in the universal jacobian over the moduli space of genus g curves. It gives rise to a normal function which is an algebro-geometric (even motivic) incarnation of the Johnson homomorphism. I will define the rank of a normal function, a measure of its non-triviality, and sketch a proof that the generic rank of the normal function of the genus g Ceresa cycle is $3g-3$, the maximum possible. This implies that the maximal power of the curvature form of the associated biextension line bundle is a volume form on the moduli space of curves of compact type and a positive measure on its Deligne–Mumford compactification. One motivation for this work comes from work of Ziyang Gao and Shou-Wu Zhang on the Arakelov theory of moduli spaces of curves.

Lecture III. Hecke actions on loops and periods of iterated Shimura integrals

Iterated Shimura integrals are iterated integrals of classical modular forms. They are elements of the coordinate ring of the relative unipotent completion of $\mathrm{SL}_2(\mathbb{Z})$, which we regard as the fundamental group of the modular curve. Francis Brown has proposed that the coordinate ring of the appropriate relative completions of $\mathrm{SL}_2(\mathbb{Z})$ generate the (conjectural) tannakian category of mixed modular motives — the category of mixed motives generated by the motives of classical modular forms.

The goal of this talk is to explain how the classical Hecke operators act on the free abelian group generated by the conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$ and, dually, on those elements of the coordinate ring of the relative completion of $\mathrm{SL}_2(\mathbb{Z})$ that are constant on conjugacy classes. The construction uses only elementary topology.

This Hecke action commutes with the natural Galois action and each Hecke operator is a morphism of mixed Hodge structure. One surprising fact is that, while the Hecke operators T_N and T_M (acting on conjugacy classes) commute when N and M are relatively prime, T_p and T_{p^2} do not commute for any prime p. Consequently, the corresponding Hecke algebra is not commutative, in contrast with the classical case.

2. Talks

Anton Alekseev – Generalized pentagon equation and the KKS coaction map.

Following Drinfeld, the Knizhnik-Zamolodchikov (KZ) associator is defined as a regularized holonomy of the KZ connection along « le droit chemin » between 0 and 1. We consider curves which are not straight, and which may have transverse self-intersections. It turns out that the corresponding regularized holonomies satisfy an analogue of the pentagon equation with extra terms corresponding to self-intersections. As an application, we compute the Kirillov-Kostant-Souriau (KKS) coaction map of a regularized holonomy, and we derive the corresponding double brackets.

The talk is based on joint works with Florian Naef and Muze Ren.

Omid Amini – Tropical algebraic cycles and tropical Hodge theory.

I will describe our current understanding of the cycle class map in the tropical setting.

First, I present the formulation of the tropical Hodge conjecture which describes the image. I sketch a proof obtained in joint work with Matthieu Piquerez.

Then, I talk about the kernel. I introduce intermediate Jacobians in the tropical setting, and define the tropical Abel-Jacobi map on homologically trivial cycles. I present few applications, in particular to the Ceresa cycle on moduli space of curves. This part is joint with Dan Corey and Leonid Monin.

Hodge theory for Kähler tropical varieties is established in joint work with Matthieu Piquerez.

Carel Faber – Arithmetic aspects of the cohomology of moduli spaces of curves.

I will present an overview of known and conjectured results about the cohomology of moduli spaces of (pointed) curves of low genus, focusing on the part of the cohomology associated to modular forms. I will also discuss some questions that seem relevant to me.

Margarida Melo – Tropicalizing moduli spaces and applications.

In the last few years, it has been understood that nice moduli spaces can be "tropicalized" via modular maps which allow to study many properties of the original spaces by looking at their tropical counterpart. Often, a first step is to give a tropical interpretation to combinatorial data used to compactify. In the talk, I will try to explain this interplay in the case of (universal) Jacobians and, time permitting, the moduli space of abelian varieties.

Florian Naef – The Kashiwara-Vergne quotient.

I will give a review of (part of) the relationship between the Kashiwara–Vergne (KV) problem and the Goldman–Turaev Lie bialgebra based on joint works with A. Alekseev, N. Kawazumi, Y. Kuno. Furthermore, I will give a characterization of the KV problem in terms of automorphisms of (certain quotients of) completed braid groups. The last part is a report on current work in progress with R. Betancourt.

Noema Nicolussi – Hybrid curves and their moduli spaces.

This talk provides an introduction to hybrid curves, a new geometric object which mixes Riemann surfaces and graphs, and their moduli spaces.

The moduli space of hybrid curves refines the classical Deligne–Mumford compactification of the moduli space of Riemann surfaces and provides a framwork to answer several analytic questions on asymptotic geometry of degenerating Riemann surfaces (e.g., asymptotics of canonical measures and Green functions). In addition, hybrid curves enjoy analogs of fundamental theorems on smooth compact Riemann surfaces, e.g. a Riemann–Roch and Abel–Jacobi theorem.

In this talk, we introduce hybrid curves and overview several recent results.

Based on joint work with Omid Amini.

Dan Petersen – A simple proof of the Mumford conjecture.

Andrea Bianchi recently gave a new proof of Mumford's conjecture on the stable rational cohomology of the moduli space of curves (first proven by Madsen and Weiss). I will explain a streamlined and simplified version of Bianchi's argument. (Joint with Ronno Das)

Ulrike Tillmann – Homology stability for generalised Hurwitz spaces and asymptotic monopoles.

Configuration spaces have played an important role in mathematics and its applications. In particular, the question of how their topology changes as the cardinality of the underlying configuration changes has been studied for some fifty years and has attracted renewed attention in the last decade.

While classically additional information is associated "locally" to the points of the configuration, there are interesting examples when this additional information is "nonlocal". With Martin Palmer we have studied homology stability in some of these cases, including Hurwitz space and moduli spaces of asymptotic monopoles.

Orsola Tommasi – Geometry of fine compactified Jacobians.

The degree d universal Jacobian parametrizes degree d line bundles on smooth curves. There are several approaches on how to extend it to a proper family over the moduli space of stable curves. In this talk, we introduce a simple definition of a fine compactified universal Jacobian, both for a single nodal curve and for families. We discuss their combinatorial characterization and explain how this leads to new examples already in the case of curves of genus 1.

This is joint work with Nicola Pagani.

Karen Vogtmann – The borders of Outer space.

An arithmetic group acts properly but not necessarily cocompactly on the associated symmetric space. Borel and Serre got around this by defining a "bordification" of the symmetric space on which the action extends to a proper cocompact action. They used their bordification to determine the virtual cohomological dimension of the arithmetic group, using the fact that the boundary of the bordification is homotopy equivalent to a spherical Tits building. J. Harer proved that mapping class groups of a punctured surface is a virtual duality group using a similar bordification of the associated Teichmuller space and the fact that the boundary of this bordification is homotopy equivalent to the curve complex, which is spherical. For the group $\mathrm{Out}(F_n)$ of outer automorphisms of a free group the analog of the symmetric space is called Outer space, and Bestvina and Feighn proved that $\mathrm{Out}(F_n)$ is a duality group by constructing an analogous bordification and proving directly that it is highly-connected at infinity. The fact that Outer space is not a manifold prevents one from using the homotopy type of the boundary to prove the duality theorem, but it is still interesting to ask how far the analogy extends, and I will address this question.

Thomas Willwacher – Cyclic model for the dg dual of the BV operad.

I describe a small cyclic homotopy operad model for the dg (Koszul-)dual of the BV operad. As an application, one can show that the Feynman transform of BV and of its dg dual are quasi-isomorphic.

The talk is based on `https://arxiv.org/abs/2311.09037`.